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Quantum localization of chaos in a ring

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Quantum localization of classical diffusion of energy for a mesoscopic metallic ring in the presence of a microwave field is studied. The condition for the importance of resonances between the electronic motion in the ring and the alternating field is considered. It is shown that this model is analogous to the one-dimensional Anderson model under certain conditions. The conditions, as well as an expression for the localization length, are obtained. A method that is based on the photonic states concept, that is valid for this case when the level width is much less than energy spacing between quasiresonant levels, is used.

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The problem of quantum localization of classical chaotic motion [1–3] is one of the central problems of quantum chaos. This phenomenon, also called dynamical localization, takes place in systems where a resonant interaction between degrees of freedom is essential and cannot be considered perturbatively.

Recently this class of problems became very relevant for the investigation of electronic motion in mesoscopic systems. The main interest in this subject is due to consideration of the localization problem in energy space of adiabatic levels for both Bloch electrons in one-dimensional systems subject to a uniform electric field [4–6] and an electron in a ring in the presence of a time dependent magnetic flux [4,5,7]. In the case of low frequency perturbation an adiabatic approach is valid. A mechanism of transitions between adiabatic levels is by Zener transition [8]. A localization length in this case is due to this transition [4]. In the case of high frequency perturbation the picture of Zener transition between adiabatic levels fails [7,9]. Transitions occur between resonant states which are not nearest neighbors. The problem of dynamical localization must be considered in the framework of theory taking into account these resonant transitions.

A purely quantum approach taking into account resonant one-photon transitions induced by an external field has been constructed to consider a localization problem for the bubble model [2]. This method uses a concept of photonic states [10,2,11] and enables one to map a problem of quantum dynamics on a one-dimensional (1D) tight-binding model. It can be instructive to apply this theory to consider the dynamical localization in mesoscopic systems, in particular in a ring, driven by high frequency perturbation. Problems related to the Anderson localization in a mesoscopic ring have been considered in [7,9,12] for high frequency perturbation (or large rate of flux change).

In this paper the dynamical localization of excitation in the energy space of electrons in a mesoscopic ring in the presence of a spatially inhomogeneous microwave (uhf) perturbation is studied. A model, where a resonant interaction between the electronic motion in the ring and the microwave field is of essential importance and leads to chaotic behavior with diffusive energy excitation in the classical limit under certain conditions, is proposed. A picture of Zener transition between adiabatic levels fails in this case [7,9]. Using a purely quantum approach in the framework of the Floquet Hamiltonian provided in Ref. [2] we show that the Floquet map for the photonic states is related to the 1D Anderson localization problem.

We shall consider one electron motion in a ring-shaped conductor of radius R. It is supposed that only a small part of the ring is exposed to the microwave wave with frequency ν. Thus, an electron of a mass m "feels" the uhf field only at every passage of this small region Δl in period 2π over the angle θ. This interaction can be considered as a kick by an effective $\delta_{2\pi}(\theta)$ potential

$$\varphi(\theta,t)=\epsilon\cos(\nu t)\sum_{n=-\infty}^{\infty}\cos(n\theta). \tag{1}$$

This interaction (1) is valid when the time of the traversal of the interaction region is less than the period of the uhf field: $t_{\Delta l}=\Delta l/v_F<2\pi/\nu$, and $\Delta l\ll R$.

Hence the Hamiltonian of the one-dimensional electron motion in the ring in the presence of the perturbation (1) reads

$$H=H_0-\epsilon V=-\frac{\hbar^2}{2J}\frac{\partial^2}{\partial\theta^2}+\epsilon\cos(\nu t)\sum_{n=-\infty}^{\infty}\cos(n\theta),\qquad J=mR^2. \tag{2}$$

Classical analysis of the system (2) predicts chaotic behavior in some finite region of phase space with diffusive excitation of the energy. The Chirikov criterion of chaos, which is very important for the following quantum analysis, can be obtained from the classical equations of motion for the angular momentum $\dot{L}=\epsilon\cos\nu t\Sigma n\sin n\theta$ and angle $\dot{\theta}=L/J$. The distance between two nearest resonances n and $n_1=n+1$ determined by the resonance condition

$$n_i\frac{L_{n_i}}{J}=\nu \tag{3}$$

has the form $\Delta L\equiv L_{n_1}-L_n=J\nu/nn_1$. The width of a separatrix is $L_{\max}=\sqrt{2J\epsilon}$. Hence the following condition:

$$K=\frac{L_{\max}}{\Delta L}=\sqrt{2J\epsilon}n_1n/J\nu\simeq\frac{2n}{\omega_c}\sqrt{\epsilon/J} \tag{4}$$

determines the condition of chaotic motion. We suppose here that the resonance number in Eq. (4) can be defined by the following expression: $n=\nu/\omega_c$, where ω_c is a constant frequency corresponding to the current electron motion in the ring with the Fermi velocity: $v_F=\omega_c R$. It is worthwhile to remark here that the conditions of both high frequency perturbation and quasi-constant-frequency for wide scale of the energy of unperturbed motion

$$n=\nu/\omega_c \gg 1, \quad \omega_c J/\hbar \gg 1, \tag{5}$$

are very important for the following quantum mechanical consideration.

Following Ref. [2], we consider the Floquet Hamiltonian

$$\hat{\mathscr{H}}=-i\hbar\frac{\partial}{\partial t}+\hat{H}_0+\epsilon V\equiv\hat{\mathscr{H}}_0+\epsilon V. \tag{6}$$

For $\epsilon=0$ the eigenfunctions of $\hat{\mathscr{H}}_0$ are given by

$$u_{j,l}=\frac{1}{2\pi}e^{il\theta}e^{i\hbar j\nu t}, \quad \hat{\mathscr{H}}_0 u_{j,l}=(E_l-j\nu)u_{j,l}, \tag{7}$$

where E_l and $e^{il\theta}$ are the eigenvalue and eigenfunctions of the unperturbed Hamiltonian $\hat{H}_0$. In this case the matrix elements of $\hat{\mathscr{H}}$ are

$$\langle u_{j,l}|\hat{\mathscr{H}}|u_{j',l'}\rangle=(E_l-\hbar j\nu)\delta_{j,j'}\delta_{l,l'}+\frac{\epsilon}{2}(\delta_{j,j'+1}+\delta_{j,j'-1}), \tag{8}$$

where we use the expression $(1/2\pi)\Sigma_n\int d\theta\, e^{i(n-l+l')\theta}=1$ for fixed l,l'.

As it follows from (3) and (5) there exists a set of resonant one-photon transitions of width of $\gamma(l)$. These resonant transitions determine a set of resonant states characterized by an integer number j with energy $E_{l_0}+\hbar\nu j$ for any unperturbed state u_{0,l_0} with energy E_{l_0}. In [2] these states were called "photonic states." Following [2] we shall denote the jth photonic state by $|l_0,j\rangle$. It is supposed that

$$1<\gamma\ll\nu. \tag{9}$$

This means that these states are not overlapped and not empty. This condition together with inequalities (5) determine conditions of validity for photonic states consideration. To determine the width γ, an analysis can be carried out in the framework of a local linear approximation for $\hat{H}_0$ [11]. In this case the linearized Floquet Hamiltonian (6) with the potential (1) corresponds to a linear kicked rotator [13]. The analysis for γ simply repeats the one carried out for photonic states width for the bubble model [11] in the linear kicked rotator approximation. For fixed energy E_{l_0} we shall express the operator $\hat{\mathscr{H}}$ in the diagonal form, using the following expansion over the photonic states:

$$u_{0,l}\equiv|u_{0,l}\rangle=\sum_j \psi_j|l_0,j\rangle. \tag{10}$$

Also, the energy E_l nearest to the energy of a photonic state can be considered as $E_l\equiv E_{l(j)}=E_{l_0}+\hbar j\nu+\Delta_j$, where Δ_j is the detuning from exact resonance [2]. Using (8) and (10), we obtain the following expression:

$$(E_{l_0}+\Delta_j)\psi_j+\frac{\epsilon}{2}(\psi_{j+1}+\psi_{j-1})=E\psi_j. \tag{11}$$

For small ϵ we can put that $E\simeq E_{l_0}+\epsilon(\chi/2)$. So, finally we obtain the equation for ψ

$$\psi_{j+1}+\psi_{j-1}+\frac{2}{\epsilon}\Delta_j\psi_j=\chi\psi_j, \tag{12}$$

where $|\chi|<1$.

Let us now pass to the detuning. It cannot be larger by modules than one half of the unperturbed level spacing $\delta_j=E_{l(j)+1}-E_{l(j)}$, otherwise it belongs to another level with $l\pm1$. We can define normalized detuning $\sigma_j=\Delta_j/\delta_j$. It follows from the quantum resonance conditions that $\sigma_j=\{\sqrt{\nu J/\hbar}j\}_{\text{frac}}$, where $\{\cdots\}_{\text{frac}}$ means fractional part. These numbers can be considered quasirandomly distributed in the interval $[-0.5,0.5]$ for $j\ll(\nu J/\hbar)$ and $\nu J/\hbar\gg1$, where $\alpha>1$. The unperturbed energy level spacing is determined approximately by $\delta_j\simeq(\hbar/J)v_F R\sim\hbar\omega_c$ and is independent of j in the framework of the considered approximation. Here the spectrum is taken to be equidistant for simplicity, which is quite reasonable for the big orbital numbers $l\sim mv_F R/\hbar=\omega J/\hbar$. In this case we obtain from (12)

$$\begin{gathered}\psi_{j+1}+\psi_{j-1}+W_j\psi_j=\chi\psi_j,\\ W_j=\frac{2\hbar\omega_c}{\epsilon}\sigma_j\equiv\Lambda\sigma_j.\end{gathered} \tag{13}$$

Equation (13) corresponds to the one-dimensional Anderson localization problem with a known exponentially localized solution for the eigenfunctions ψ_j. This problem has been considered both analytically and numerically [14–16]. In the both cases of small ($\Lambda\ll1$) and large ($\chi\ll\Lambda$) "disorder" (SD and LD, respectively) the localization length ξ was found for $\chi=1$ and $j\to\infty$ [15,16]:

$$\xi_\infty^{-1}\simeq\begin{cases}\xi_{\text{SD}}=\dfrac{\Lambda^2}{72}, & \chi\gg\Lambda\\ \xi_{\text{LD}}=\ln(\Lambda/2)-1, & \chi\ll\Lambda.\end{cases} \tag{14}$$

Unlike [14–16], the number of energy levels j, where the localization takes place, is restricted by the resonant conditions (3) with $n\gg1$. This means that $j<j_{\max}=\omega_c J/\hbar$. The rate of decay is measured by the Lyapunov exponent $(\xi_{j_{\max}})^{-1}$, and is evaluated numerically by the transfer matrix method [14]. The numerical analysis of ξ indicates a localization process except in the case of small disorder, when $\Lambda\ll1$. In this case the localization length exceeds the energy range, where the photonic states are $\xi>j_{\max}$ for any eigenvalues $\chi<1$ and $j_{\max}\sim10^3$. An exponential fitting of the eigenfunctions ψ_j is meaningful when the following condition holds: $1\ll\xi\ll j_{\max}$. This regime is found as well for "moderate disorder," when $\chi<\Lambda\sim1$ and $\xi\sim\xi_{\text{SD}}$. It follows from (4) and (14) that the condition of the validity of the solution for localization length reads

$$1 \ll \frac{j_{\max}}{\xi} = \frac{8}{9}\left(\frac{\omega_c J}{\hbar}\right)^{-1} \frac{n^4}{K^4}. \tag{15}$$

In conclusion, the constructed model with Hamiltonian (2) exhibits a localization in energy space, which is described by the Anderson model. This result is obtained purely by the quantum approach of Ref. [2], which is valid here for some restriction determined by the expressions (5), (9), and (15). For describing an experimental realization it may be important to take impurities in the ring into account. An influence of the impurities on the localization length can be studied in the framework of the Hamiltonian (2) as well.

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